On the algebraic and smooth structures of the diffeomorphism group of compact manifold

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Abstract. The title seems more pretentious than it was meant to be. In this mini-course we will just introduce some notions of infinite dimensional Lie groups as well as their topology and representations. We will focus mainly on the natural action of the diffeomorphism group $\mathrm{Diff}(M)$ on a compact manifold M as well as on its right action by pullback on the bundle of symmetric covariant two tensors on M. Groups of diffeomorphisms of a manifold M have many of the properties of finite dimensional Lie groups, but also differ in many ways, for example, the exponential mappings are defined but are not locally surjective or injective. There are groups of diffeomorphisms which are smooth manifolds but only right translations are smooth. Surprisingly, the algebraic structure of the identity component $\mathrm{Diff}_0(M)$ determines the smooth structure of M. If $\mathrm{Diff}_0(M)$ is isomorphic to $\mathrm{Diff}_0(N)$, then M is diffeomorphic to N. Our interest in the present topic is motivated by the problem of finding a metric on a compact manifold with prescribed isometry group. We will show a theorem, due to Jean Pierre Bourguignon, that says that if G is a compact Lie subgroup of the group of diffeomorphism of the manifold satisfying a given algebraic property then there exists a metric g such that the isometry group is G. Motivated by this result we are currently investigating the related problem of finding a metric on homogeneous spaces with some more geometric or algebraic information prescribed. The prototype cases are compact Lie groups equipped with bi-invariant metric or more generally symmetric spaces.

This minicourse is organized as follows. We will start with a quick review of manifolds, Lie groups, G-action on manifolds, and we will finish this initial part with the Slice Theorem. In the second part, we will give the definition of a Hilbert Lie group as well as other weaker notions of infinite dimensional Lie group and some of their properties. The main goal of this part will be to show an analog of the Slice Theorem for $\mathrm{Diff}(M)$-action on the space of smooth Riemannian metrics of M. In the last part, we will show some applications to the problem of finding a Riemannian metric with prescribed isometry group.

References

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